Available online at www.sciencedirect.com
SCIENCE DIRECT®

Journal of Pure and Applied Algebra 194 (2004) 273–279

JOURNAL OF PURE AND APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

On the recognition problem of quasi-homogeneous locally nilpotent derivations in dimension three

Arno van den Essen*, Daan Holtackers

Department of Mathematics, University of van Nijmegen, Toernooiveld, 6525 ED Nijmegen, The Netherlands

Received 2 May 2003; received in revised form 18 March 2004
Communicated by C.A. Weibel
Available online 20 July 2004

Abstract

We give a criterion to decide if a given w-homogeneous derivation on $A := k[X_1, X_2, X_3]$ is locally nilpotent. We deduce an algorithm which decides if a k-subalgebra of A, which is finitely generated by w-homogeneous elements, is the kernel of some locally nilpotent derivation.

MSC: 13N15; 14R20; 14R10

0. Introduction

Let $k[X_1, \ldots, X_n]$ be the polynomial ring in n variables over a field k of characteristic zero. A k-derivation D on this ring is called locally nilpotent if for each a in $k[X_1, \ldots, X_n]$ there exists a positive integer m such that $D^m(a) = 0$. Locally nilpotent derivations play a crucial role in the study of various problems in affine algebraic geometry (see [7] for many examples). Nevertheless, our understanding of these derivations is very limited; only the case $n=2$ is well-understood thanks to a classical result due to Rentschler in [11]. However, if $n=3$ (and more generally $n \geqslant 3$) there are many questions unanswered. For example, we do not have a complete description of all locally nilpotent derivations on $k[X_1, X_2, X_3]$ (the classification problem) nor do we have an

* Corresponding author. Fax: +31-024-3652140.
E-mail addresses: essen@math.kun.nl (A. van den Essen), holtacke@sci.kun.nl (D. Holtackers).

0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2004.05.003

algorithm which decides if a given derivation on $k[X_1,X_2,X_3]$ is locally nilpotent (the recognition problem). Another open problem is the sub-algebra problem which asks for a criterion/algorithm to decide if a given k-subalgebra of $k[X_1,X_2,X_3]$ can occur as the kernel of some locally nilpotent derivation.

On the other hand it is known that the kernel of a non-zero locally nilpotent derivation on $k[X_1,X_2,X_3]$, denoted by $k[X_1,X_2,X_3]^D$, is generated by two algebraically independent elements; this result is due to Miyanishi in [10]. In case D is weighted homogeneous these two generators can be chosen to be homogeneous. This result was obtained by Zurkowski in 1993 (see [12]). More recently in a series of highly non-trivial papers [2–4] Daigle and Russell obtain a geometric description of all weighted homogeneous locally nilpotent derivations on $k[X_1,X_2,X_3]$ i.e. they solved the classification problem for these derivations. In this paper, we solve both the recognition and the sub-algebra problem for quasi-homogeneous derivations on $k[X_1,X_2,X_3]$.

1. Preliminaries

Throughout this paper, k will denote a field of characteristic zero and $A := k[X_1,\dots,X_n]$. If R is a k-subalgebra of A then $Q(R)$ denotes the quotient field of R.

The following proposition summarizes some results concerning locally nilpotent derivations which we will need in the sequel.

Proposition 1. *Let D be a non-zero locally nilpotent derivation on A. Then*

(i) *$tr\,deg_k\, Q(A^D) = n-1$;*
(ii) *let $a \in A$ with $D(a) \neq 0$ and denote the minimal power of D that annihilates a by $\rho(D,a)$, then $\rho(D,a) = [Q(A) : Q(A^D)(a)] + 1$;*
(iii) *let $f_1,\dots,f_{n-1}$ be algebraically independent elements of A^D and denote by D_1 the Jacobian derivation defined by*

$$D_1(h) := det\,J(f_1,\dots,f_{n-1},h), \quad for\ all\ h \in A.$$

Then there exist non-zero elements a and b in A^D such that $aD = bD_1$. Furthermore D_1 is locally nilpotent.

The results (i) and (ii) can be found in [7, 1.3.32] and (iii) in [8].

A consequence of (ii) is the so-called partial nilpotency criterion (see [5] or [7, 1.4.17]).

Partial nilpotency criterion: Let D be a non-zero k-derivation on A and suppose we can find $f_1,\dots,f_{n-1}$ in A^D algebraically independent over k. Put

$$N := \max_i \{[Q(A) : k(f_1,\dots,f_{n-1})(X_i)] \mid D(X_i) \neq 0\}.$$

Then N is finite and D is locally nilpotent iff $D^{N+1}(X_i) = 0$ for all i.

In dimension 3, we have the following more precise statement:

Proposition 2. *Let D be a non-zero locally nilpotent derivation on A. Then there exist f, g algebraically independent over k such that $A^D = k[f, g]$. Furthermore, if $gcd_i\, D(X_i) = 1$, then $D = \lambda D_1$ for some $\lambda \in k^*$, where $D_1(h) := det\, J(f, g, h)$ for all $h \in A$.*

As observed earlier, the first part of this proposition is due to Miyanishi in [10]. The second part is due to Daigle in [1].

For the reader's convenience we recall some facts concerning quasi-homogeneous polynomials and derivations.

Now let $n = 3$ and $w := (w_1, w_2, w_3) \in (\mathbb{N}^+)^3$. We make A into an $\mathbb{N}$-graded ring by defining A_m to be the k-vectorspace generated by all monomials $X^\alpha := X_1^{\alpha_1} X_2^{\alpha_2} X_3^{\alpha_3}$ with w-degree equal to m i.e. $\langle w, \alpha \rangle := w_1\alpha_1 + w_2\alpha_2 + w_3\alpha_3 = m$. A derivation D on A is called w-homogeneous with degree $w(D)$ if $D(A_m) \subset A_{m+w(D)}$ for all $m \geqslant 0$. In case D is a locally nilpotent w-homogeneous derivation on A another proof of the first part of Proposition 2 was given by Zurkowski in [12]. More precisely,

Proposition 3. *If D is a non-zero w-homogeneous locally nilpotent derivation on A, then there exist w-homogeneous elements f and g in A^D such that $A^D = k[f, g]$.*

To conclude this section we give a consequence of Bezout's theorem.

Proposition 4. *Let $f, g \in k[X, Y]$ be algebraically independent over k. Then $[k(X, Y) : k(f, g)] \leqslant deg\, f \cdot deg\, g$.*

Proof. Let $\bar{k}$ be an algebraic closure of k. Since $[\bar{k}(X, Y) : \bar{k}(f, g)] = [k(X, Y) : k(f, g)]$ we may replace k by $\bar{k}$ and hence assume that k is algebraically closed. Let $F = (f, g) : k^2 \to k^2$ be the corresponding polynomial map. Since f, g are algebraically independent over k the map F is dominant. So for almost all $\lambda = (\lambda_1, \lambda_2) \in k^2$, $\#F^{-1}(\lambda) = d(F) := [k(X, Y) : k(f, g)]$ (see [7, B.2.1]). Choose such a $\lambda = (\lambda_1, \lambda_2)$. Then $d(F) = \#F^{-1}(\lambda)$ equals the number of common zeroes of the system $f - \lambda_1 = 0$, $g - \lambda_2 = 0$. So by Bezout's theorem we get $d(F) \leqslant deg(f - \lambda_1) \cdot deg(g - \lambda_2) = deg\, f \cdot deg\, g$. $\square$

2. A bound for the nilpotency index of a locally nilpotent derivation

Throughout this section $A := k[X_1, X_2, X_3]$ and $D = a_1\partial_1 + a_2\partial_2 + a_3\partial_3$ is a non-zero derivation on A; so $a_i = D(X_i) \in A$ for all i. Furthermore, we fix $w = (w_1, w_2, w_3) \in (\mathbb{N}^+)^3$ and assume that D is w-homogeneous of degree $w(D)$. Finally, put $|w| := w_1 + w_2 + w_3$ and $w_{min} := min_i(w(X_i))$.

If D is locally nilpotent there exists a positive integer m such that $D^m(X_i) = 0$ for all i. The smallest such an integer is called the nilpotency index of D. The next result, which is the main theorem of this paper, gives an upperbound for this nilpotency index. For a real number x, $[x]$ denotes the largest integer smaller than x.

Theorem 1. *Let* $N := [w_{min}^{-2}((w(D)+|w|)/2)^2]$ *and assume* $gcd(a_1,a_2,a_3)=1$. *If* D *is locally nilpotent, then* $D^{N+1}(X_i)=0$ *for all* i.

Proof. Replacing k by $\bar{k}$ we may assume that k is algebraically closed. By Proposition 3, $A^D = k[f,g]$ for some algebraically independent w-homogeneous elements f and g and by Proposition 2, $D = \lambda\, det\, J(f,g,-)$ for some $\lambda \in k^*$. Of course we may assume that $\lambda = 1$. So

$$\begin{aligned} D &= det\, J(f,g,-) \\ &= (f_{X_2}g_{X_3} - f_{X_3}g_{X_2})\partial_1 - (f_{X_1}g_{X_3} - f_{X_3}g_{X_1})\partial_2 + (f_{X_1}g_{X_2} - f_{X_2}g_{X_1})\partial_3. \end{aligned}$$

Now let $D(X_i) \neq 0$. Then by the partial nilpotency criterion we obtain that

$$\rho(D,X_i) = [k(X_1,X_2,X_3) : k(f,g)(X_i)] + 1.$$

So it suffices to prove that

$$[k(X_1,X_2,X_3) : k(f,g)(X_i)] \leqslant N. \tag{1}$$

To see (1) observe that f, g and X_i are algebraically independent over k for otherwise $det\, J(f,g,X_i) = 0$ i.e. $D(X_i) = 0$, contradiction. So by Proposition 4 applied to $k(X_i)$ (instead of k) we get

$$[k(X_1,X_2,X_3) : k(f,g)(X_i)] \leqslant deg\, f \cdot deg\, g.$$

So, in order to prove (1) it suffices to prove

$$deg\, f \cdot deg\, g \leqslant N. \tag{2}$$

Therefore put $d_1 := deg_w\, f$ and $d_2 := deg_w\, g$. Since D is w-homogeneous of degree $w(D)$ it follows, looking at the coefficients of the ∂_i in the first equation of the proof, that

$$deg_w\, f + deg_w\, g - deg_w\, X_2 - deg_w\, X_3 = deg_w\, D(X_1) = deg_w\, X_1 + w(D),$$

and thus

$$deg_w\, f + deg_w\, g - |w| = w(D),$$

hence

$$d_1 + d_2 = |w| + w(D). \tag{3}$$

Consequently, $d_1 d_2 = d_1(-d_1 + |w| + w(D)) = -d_1^2 + (|w| + w(D))d_1$. The quadratic polynomial in d_1 has its maximum at $d_1 = \frac{1}{2}(|w| + w(D))$ and this maximum equals $(\frac{1}{2}(|w| + w(D)))^2$. Consequently

$$d_1 d_2 \leqslant \left(\frac{w(D)+|w|}{2}\right)^2. \tag{4}$$

Since obviously $deg\, f \leqslant 1/w_{min}\, deg_w\, f = d_1/d_{min}$ and similarly $deg\, g \leqslant d_2/w_{min}$, it follows from (4) that

$$deg\, f \cdot deg\, g \leqslant \left[w_{min}^{-2} \left(\frac{w(D)+|w|}{2}\right)^2 \right] = N$$

which proves (2) and completes the proof of the theorem. □

Corollary 1. *Let N be as in* 2.1 *and* $gcd(a_1, a_2, a_3) = 1$. *Then D is locally nilpotent iff $D^{N+1}(X_i) = 0$ for all i.*

3. Applications

In this section, we show how Corollary 1 can be used to solve both the recognition problem and the sub-algebra problem for $A := k[X_1, X_2, X_3]$.

3.1. Solution of the recognition problem

Proposition 5. *We have a criterion to decide if a given w-homogeneous derivation on A is locally nilpotent.*

Proof. Let $a := gcd(a_1, a_2, a_3)$. Then $D = a\bar{D}$ for some k-derivation $\bar{D}$ on A. Observe that $gcd_i \bar{D}(X_i) = 1$ and that both a and $\bar{D}$ are w-homogeneous. Furthermore, by [7], 1.3.34 D is locally nilpotent iff $\bar{D}(a) = 0$ and $\bar{D}$ is locally nilpotent. Since $\bar{D}$ satisfies the hypothesis of Corollary 1 we can verify if $\bar{D}$ (and hence D) is locally nilpotent. □

3.2. Solution of the sub-algebra problem

To solve the sub-algebra problem we need the following result:

Proposition 6. *Let $R := k[f_1, \ldots, f_r] \subset A$ be a finitely generated k-subalgebra of A, generated by w-homogeneous elements f_i of A. Assume that f_1 and f_2 are algebraically independent over k.* Put $D_0 := det\, J(f_1, f_2, -)$. *Then R is the kernel of some non-zero locally nilpotent derivation D on A iff D_0 is locally nilpotent and $R = A^{D_0}$.*

Proof. Assume $R = A^D$ for some non-zero locally nilpotent derivation D on A. Since $f_1, f_2 \in R$ are algebraically independent over k, it follows from Proposition 1 (iii) that there exist non-zero elements a and b in A^D such that $aD = bD_0$ and that D_0 is locally nilpotent. From the equality $aD = bD_0$ it follows that $A^D = A^{D_0}$, so $A^{D_0} = R$. □

Proposition 7. *There is a solution of the sub-algebra problem.*

Proof. (i) We may assume $f_1 \notin k$. First check if for some $i \geqslant 2$ f_1 and f_i are algebraically independent over k; (if this is not the case then R cannot be the kernel of some locally nilpotent derivation on A by Proposition 1 (i)). To check if f_1 and f_i are algebraically independent over k one just has to verify if $det\, J(f_1, f_i, -)$ is non-zero.

(ii) Interchanging f_2 and f_i if necessary we may assume that f_1 and f_2 are algebraically independent over k. Then in order to see if R is the kernel of some locally nilpotent derivation on A we need, according to Proposition 6, to verify that $D_0 := det\, J(f_1, f_2, -)$ is locally nilpotent, which can be done by Proposition 5 since obviously D_0 is w-homogeneous, and also we need to verify if $R = A^{D_0}$.

(iii) To check this last equality we must compute A^{D_0}. This can be done by the kernel algorithm given in [6] (see also [7, 1.4]). However, a much faster way is to use Maubach's algorithm given in [9], namely we know that there exist w-homogeneous elements f and g with $A^{D_0} = k[f,g]$ and by (3) these elements satisfy $w - deg\, f + w - deg\, g \leqslant w(D_0) + |w|$. So in order to find generators of A^{D_0} one only has to compute all elements of A^{D_0} uptill w-degree $w(D_0) + |w|$. This can be done very fast by the Maubach algorithm. Furthermore, this algorithm automatically gives the minimal number of generators of A^{D_0} i.e. we effectively find w-homogeneous elements f and g with $A^{D_0} = k[f,g]$!

(iv) Once A^{D_0} is computed i.e. f and g are found such that $A^{D_0} = k[f,g]$ one can use the algebra membership algorithm (see [7, Proposition C.2.3]) to verify if $k[f_1,\ldots,f_r]$ and $k[f,g]$ are equal. $\square$

4. Final remarks

If we consider arbitrary non-homogeneous derivations on A and want to obtain a result similar to Theorem 1, the situation is much more complicated. Again, using Proposition 2, we may assume that $D = det\, J(f,g,-)$ for some (unknown) algebraically independent elements f and g. Suppose for example that $D(X_3) \neq 0$. In order to have an upperbound for $\rho(D, X_3)$ we would like to have an estimate for $[k(X_1,X_2,X_3) : k(f,g,X_3)]$ in terms of the degrees of the elements $D(X_i)$. The following example shows that this is not an easy job.

Example 1. Let $f,g \in k[X_1,X_2]$ with $det\, J(f,g) = 1$. Put $D := det\, J(f,g,-)$ (where we view f and g as polynomials in A). So in fact $D = \partial_3$ and hence the maximum of the degrees of the elements $D(X_i)$ is obviously equal to 0. On the other hand,

$$[k(X_1,X_2,X_3) : k(f,g,X_3)] = [k(X_1,X_2) : k(f,g)].$$

However, we do not know of any estimate of this degree (unless of course we know that the two-dimensional Jacobian conjecture is true, for then $[k(X_1,X_2) : k(f,g)] = 1$). This leads to the following question.

Question 1. Does there exist a positive integer N_0 such that $[k(X_1,X_2) : k(f,g)] \leqslant N_0$ for all pairs f,g in $k[X_1,X_2]$ satisfying the Jacobian condition $det\, J(f,g) = 1$?

4.1. Checking that a derivation is not locally nilpotent

Inspite of the fact that we are far from a complete solution of the three-dimensional recognition problem, the homogeneous result described in proposition 5 can very often be used to show that a given derivation D on A is not locally nilpotent.

The idea is the following: choose a "clever" w-grading on A and write D as a finite sum of w-homogeneous derivations, say $D = D_{p_1} + \cdots + D_{p_m}$ where $p_1 < \cdots < p_m$ and each D_i is w-homogeneous of degree i. Now use the fact that if D is locally nilpotent, then so is D_{p_m} (see [7, 5.1.13]). Consequently, if D_{p_m} is not locally nilpotent (what we can verify using Proposition 5), then D is not locally nilpotent either!

To conclude this paper we give an extension of Proposition 5.

Proposition 8. *Let D be a non-zero derivation on A. Let $w=(w_1,w_2,w_3)\in(\mathbb{N}^+)^3$ and suppose that $D=D_{p_1}+\cdots+D_{p_m}$ (as before) a sum of w-homogeneous derivations. If D_i and D_j commute for all i,j, then it is possible to decide if D is locally nilpotent.*

Proof. An immediate consequence of Proposition 5 and Lemma 1 below. □

Lemma 1. *Situation as in Proposition 8. Then D is locally nilpotent iff D_i is locally nilpotent for all i.*

Proof. ($\Leftarrow$) Since the D_i's commute we have for each $N\geqslant 1$ that $(D_{p_1}+\cdots+D_{p_m})^{mN}$ is a finite sum of monomials $D_{p_1}^{i_1}\cdots D_{p_m}^{i_m}$ with coefficients in $\mathbb{N}^+$ where at least one of the $i_j\geqslant N$. Now let $a\in A$. Choose N such that $D_i^N(a)=0$ for all i. Then $D^{mN}(a)=0$.

($\Rightarrow$) Let $1\leqslant i\leqslant 3$. Since, D is locally nilpotent $D^N(X_i)=0$ for some $N\geqslant 1$. Looking at the highest w-homogeneous part it follows that $D_{p_m}^N(X_i)=0$. It follows that D_{p_m} is locally nilpotent. Since D and D_{p_m} commute we get that $D-D_{p_m}$ is also locally nilpotent. Repeating this argument we find that all D_i are locally nilpotent. □

Acknowledgements

The authors would like to thank the referee for his constructive report.

References

[1] D. Daigle, On some properties of locally nilpotent derivations, J. Pure Appl. Algebra 114 (1997) 221–230.
[2] D. Daigle, Homogeneous locally nilpotent derivations of $k[X,Y,Z]$, J. Pure Appl. Algebra 128 (1998) 109–132.
[3] D. Daigle, P. Russell, Affine rulings of normal rational surfaces, Osaka J. Math. 38 (2001) 37–100.
[4] D. Daigle, P. Russell, On weighted projective planes and their affine rulings, Osaka J. Math. 38 (2001) 101–150.
[5] A. van den Essen, Locally nilpotent derivations and their applications III, J. Pure Appl. Algebra 98 (1993) 15–23.
[6] A. van den Essen, An algorithm to compute the invariant ring of a $\mathscr{G}_a$-action on an affine variety, J. Symbolic Comput. 16 (1993) 531–555.
[7] A. van den Essen, Polynomial automorphisms and the Jacobian Conjecture, Progress in Mathematics, Vol. 190, Birkhäuser, Verlag, Basel, Boston, Berlin, 2000.
[8] L. Makar-Limanov, Locally Nilpotent Derivations, A New Ring Invariant and Applications, Lecture Notes, 1998, (unpublished).
[9] S. Maubach, An algorithm to compute the kernel of a derivation up to a certain degree, J. Symbolic Comput. 29 (2000) 959–970.
[10] M. Miyanishi, Normal affine subalgebras of a polynomial ring, algebraic and topological theories—to the memory of Dr. Takehiko Miyata, Kinokuniya, Tokyo, 1985, pp. 37–51.
[11] R. Rentschler, Opérations du groupe additif sur le plan, C. R. Acad. Sci. Paris 267 (1968) 384–387.
[12] V.D. Zurkowski, Locally finite derivations, preprint 1993 (unpublished).